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A GAME-THEORETIC SOLUTION TO AN ENGAGEMENT PROBLEM

Walter O. Egerland

January 1979

US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
BALLISTIC RESEARCH LABORATORY
ABERDEEN PROVING GROUND, MARYLAND

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A simple model - termed "fire control" by one player and "survivability" by the other - is formulated and solved as a mathematical zero-sum game of continuous type on the unit square.

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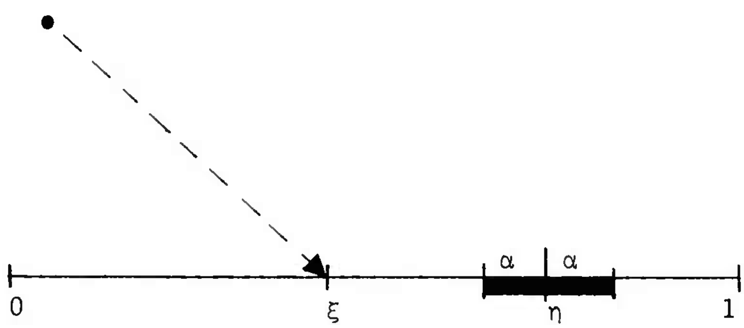
I. INTRODUCTION.

With the advent of more accurate burst fire weapons such as the Antiarmor Automatic Cannon, the question of how to program a sequence of rounds against a target of finite extent is raised. Probably the most basic problem in this regard is solved here on a game-theoretic basis. The formulation of the game in Section II. originated with Dr. S. S. Wolff of BMD, who also correctly guessed at its value and the nature of an especially simple set of optimal strategies. This report is the first account of a continuing effort to reassess the applicability of game-theoretic principles to problems in fire control and survivability. It is only recently that positive expectations have been attached again to endeavors in that field.

II. FORMULATION OF THE GAME.

Consider the following two-person zero-sum game: Player X and Player Y independently and simultaneously, each according to his own rule (or "strategy"), choose a point on the closed unit interval [0,1]; if, when the two chosen points are revealed, they are no farther apart than a pre-selected distance $\alpha (0 < \alpha < 1/2)$, Player Y pays Player X one unit, otherwise he pays (and X receives) nothing. For motivational purposes, we may think of the unit interval as a scaling of the possible locations of a maneuvering or evasive target (Player Y) during one time-of-flight of a projectile fired by Player X. A "play of the game" consists in X choosing an aim point and Y choosing a location to maneuver to; the distance α is Y's (scaled) half-width (Figure 1).



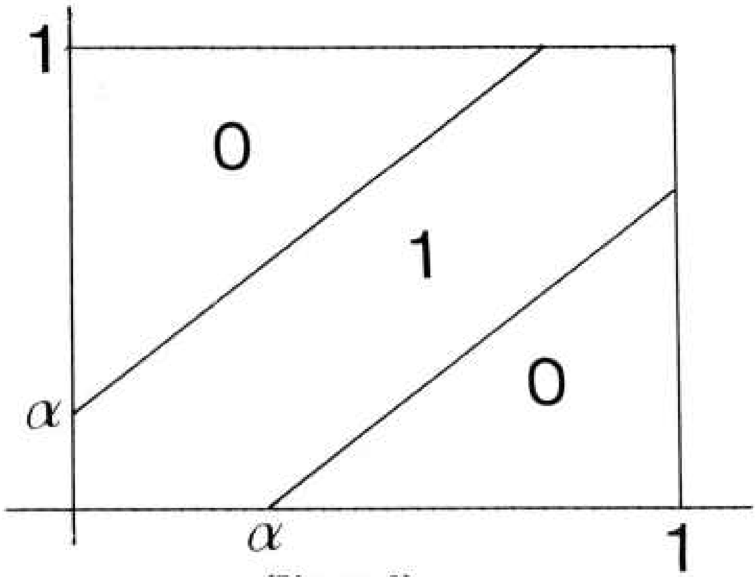
(Figure 1)

This simple model - termed "fire control" by one player and "survivability" by the other - has a natural formulation as a mathematical game of continuous type on the unit square. Let X choose his aim point ξ according to the probability distribution $x(\cdot)$; let Y choose his location η by the distribution $y(\cdot)$; define the payoff function, or kernel, by (Figure 2)

$$K(\xi,\eta) = \begin{cases} 1 & |\xi - \eta| \leq \alpha \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Then the expected payoff in one play is

$$V(x,y) = \int_0^1 \int_0^1 K(\xi,\eta)dx(\xi)dy(\eta). \tag{2}$$



(Figure 2)

Due to the discontinuity of the kernel along the lines $\eta = \xi \pm \alpha$, the existence of optimal strategies for the game is not a consequence of a general existence theorem. Nevertheless, in Section III it is shown that the game has a value in the sense that

$$\sup_x \inf_y V(x,y) = \inf_y \sup_x V(x,y).$$

Furthermore, the value depends in a transparent fashion on the parameter α. Thus a proper interpretation of the solution of the game and delimitation of a rational mode of behavior for each of the antagonists is assured.

III. PROOF FOR EXISTENCE OF A VALUE; OPTIMAL STRATEGIES.

We begin by noting that the existence of a value for the game defined by (1) and (2) of Section II is equivalent to the existence of a pair of strategies x^o and y^o satisfying the inequalities

$$V(x,y^o) \leq V(x^o,y^o) \leq V(x^o,y)$$

for all strategies x and y, respectively. Thus x^o maximizes $V(x,y^o)$ while y^o minimizes $V(x^o,y)$. The problem is, therefore, to find x^o and y^o such that

$$\sup_x \int_o^1 \int_o^1 K(\xi,\eta)dx(\xi)dy^o(\eta) = \inf_y \int_o^1 \int_o^1 K(\xi,\eta)dx^o(\xi)dy(\eta).$$

We derive first two expressions for $V(x,y)$ and develop an inequality that suggests what type of optimal strategies we search for. In order to allow intuitive considerations to be fully reflected in the formalism with a minimum of technical interference, it is of advantage to set

$$K(\xi,\eta) = 1 - k(\xi,\eta).$$

The expected payoff (2) then becomes

$$V(x,y) = 1 - \int_o^1 \int_o^1 k(\xi,\eta)dx(\xi)dy(\eta),$$

where (Figure 3)

$$k(\xi,\eta) = \begin{cases} 1 & |\xi - \eta| > \alpha \\ 0 & \text{otherwise.} \end{cases}$$

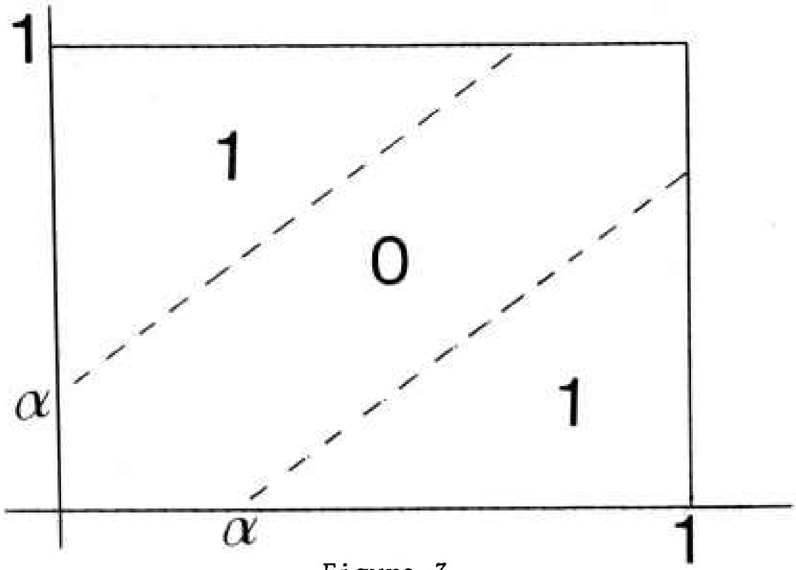


Figure 3

Carrying out one integration, we obtain for V(x,y) the equivalent expressions

(3) $$V(x,y) = 1 - \int_{[0,1-\alpha)} \big(1 - x(\eta+\alpha)\big)\, dy(\eta) - \int_{(\alpha,1]} x[(\eta-\alpha)^-]\, dy(\eta)$$

and

(4) $$V(x,y) = 1 - \int_{[0,1-\alpha)} \big(1 - y(\xi+\alpha)\big)\, dx(\xi) - \int_{(\alpha,1]} y[(\xi-\alpha)^-]\, dx(\xi),$$

in which $x(\xi^-)$ denotes $x(\xi^-) = \lim_{t\to\xi^-} x(t)$.

From (4) we have

$$V(x,y) \leq 1 - \inf_{[0,1-\alpha)} \big(1-y(\xi+\alpha)\big) x[(1-\alpha)^-] - \inf_{(\alpha,1]} y[(\xi-\alpha)^-]\big(1-x(\alpha)\big) \; .$$

Evaluating the infima on the right side, we obtain the inequality

(5) $$V(x,y) \leq 1 - \big(1-y(1^-)\big)\, x[(1-\alpha)^-] - y(0)\big(1 - x(\alpha)\big) \; ,$$

which is valid for all x. Clearly, Y wants $y(1^-) \neq 1$ and $y(0) \neq 0$ because he wants to minimize. Hence, if Y gives both considerations equal weight and adopts the mixed strategy

$$y^* = \frac{1}{2} I_0 + \frac{1}{2} I_1,$$

it follows from (3) that for all x

$$V(x,y^*) = \frac{1}{2} - \frac{1}{2}\left\{x[(1-\alpha)^-] - x(\alpha)\right\},$$

which does not exceed 1/2 (for all $\alpha < 1/2$) because of the monotonicity of x. Against y^* player X can maximize by choosing $x(\xi)$ = constant in the interval $\alpha \leq \xi < 1 - \alpha$, e.g., by adopting the mixed strategy

$$x^* = \frac{1}{2} I_\alpha + \frac{1}{2} I_{1-\alpha},$$

in which case his expected return becomes exactly 1/2. If on the other hand, X plays x^*, we have from (4)

$$V(x^*,y) = \frac{1}{2} + \frac{1}{2}\left(y(2\alpha) - y[(1 - 2\alpha)^-]\right),$$

which exceeds or equals 1/2 if $\alpha \geq 1/4$, and becomes exactly 1/2 if Y chooses $y(\eta)$ = constant in the interval $(1-2\alpha) < \eta \leq 2\alpha$. In particular, he may choose y^* to minimize $V(x^*,y)$.

Summarizing, we have shown that for target half-widths α from the interval $1/4 \leq \alpha < 1/2$ the strategies x^* and y^* are optimal, and that the value v of the game in this case is $v = 1/2$.

For smaller target half-widths we arrive at a generalization of the inequality (5) by appropriate subdivision of the integration intervals $[0,1-\alpha)$ and $(\alpha,1]$ in (4). The essential result of this experimentation, the details of which are omitted, leads to the following <u>assertion</u>:

If for a given target half-width α the positive integer n is such that

$$1/2n \leq \alpha < 1/2(n-1), \quad n = 2, 3, \ldots$$

the strategies

$$(6)\qquad x^o = \frac{1}{n}\sum_{j=0}^{n} I_{\mu_j}, \qquad \mu_j = \frac{2j-1}{2n},$$

and

$$(7)\qquad y^o = \frac{1}{n}\sum_{j=0}^{n-1} I_{\nu_j}, \qquad \nu_j = \frac{j}{n-1},$$

are optimal strategies for X and Y, respectively, and

$$v = \frac{1}{n}$$

is the value of the game.

To prove the assertion that y^o is optimal, we have from (4) and (6)

$$V(x^o,y) = 1 - \frac{1}{n}\sum_{j=1}^{n-1}\left(1-y(\mu_j+\alpha)\right) - \frac{1}{n}\sum_{j=2}^{n} y[(\mu_j-\alpha)^-]$$

$$= \frac{1}{n} + \frac{1}{n}\sum_{j=1}^{n-1}\left\{y(\mu_j+\alpha) - y[(\mu_{j+1}-\alpha)^-]\right\}.$$

Since $\mu_j - \mu_{j+1} + 2\alpha \geqq 0$, the sum on the right is non-negative for all y, and therefore $V(x^o,y) \geqq \frac{1}{n}$. Since also

$$\nu_{j-1} < \mu_{j+1} - \alpha < \mu_j + \alpha < \nu_j$$

for $n = 2, 3, \ldots,$ and $j = 1, 2, \ldots, n-1$, we have from (7)

$$y^o(\mu_j+\alpha) = y^o[(\mu_{j+1}-\alpha)^-].$$

Hence,

$$\inf_y V(x^o,y) = V(x^o,y^o) = \frac{1}{n} .$$

Similarly, to show that x^o is optimal, we obtain from (3) and (7)

$$V(x,y^o) = 1 - \frac{1}{n}\sum_{j=0}^{n-2}\Big(1 - x(\nu_j+\alpha)\Big) - \frac{1}{n}\sum_{j=1}^{n-1} x[(\nu_j-\alpha)^-]$$

$$= \frac{1}{n} - \frac{1}{n}\sum_{j=0}^{n-2}\Big\{x[(\nu_{j+1}-\alpha)^-] - x(\nu_j+\alpha)\Big\} .$$

Since $\nu_{j+1} - \nu_j - 2\alpha \geqq 0$, the sum on the right is non-negative for all x, so that $V(x,y^o) \leqq 1/n$. Since also

$$\mu_{j+1} \leqq \nu_{j+\alpha} < \nu_{j+1} - \alpha \leqq \mu_{j+2}$$

for $n = 2, 3, \ldots.,$ and $j = 0, 1, \ldots., n - 2$, it follows from (6) that

$$x^o[(\nu_{j+1}-\alpha)^-] = x^o(\nu_j+\alpha).$$

Hence,

$$\sup_x V(x,y^o) = V(x^o,y^o) = \frac{1}{n} .$$

This completes the proof of the assertion. Figure 4 illustrates the optimal strategies (6) and (7) for the cases $n = 2, 3, 4, 5,$ and 6. Arrows ($\downarrow$) indicate aim points of the gunner (Player X), and points ($\cdot$) indicate the centers of the evasive target (Player Y), all to be chosen with equal probability $1/n$. The half-width of the target was chosen as $\alpha = 1/2n$.

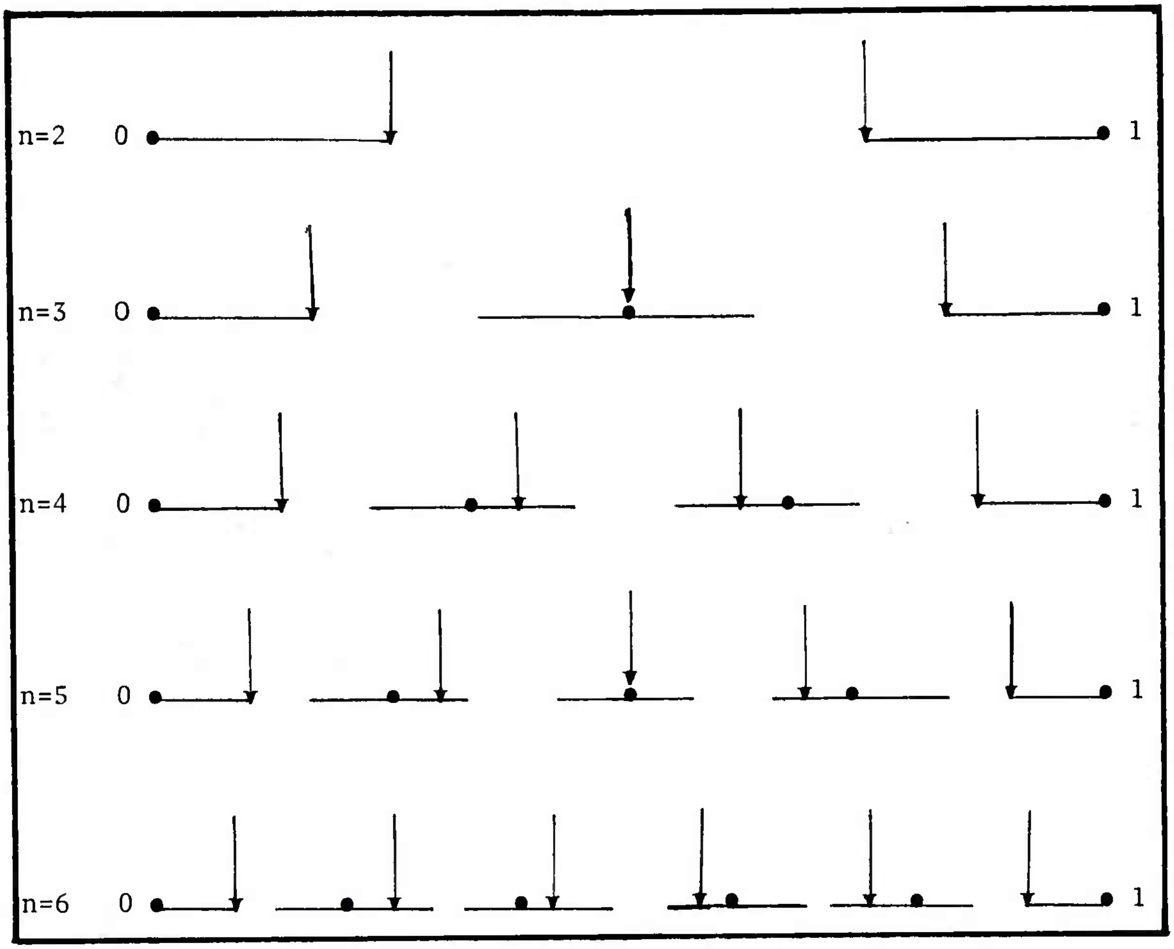


Figure 4

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Karlin, S., Mathematical Methods and Theory in Games, Programming, and Economics, Vol.II (The Theory of Infinite Games), Addison-Wesley Publishing Company, Inc., 1959.

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